Functional Analysis Lecture Notes (2024/2025)

Griffin Reimerink

Contents

1 Linear spaces

1.1 Linear spaces

Definition *Linear space*

A **linear space** X over a field $\mathbb{K}$ is a set with two operations:

- Addition: $x, y \in X \implies x + y \in X$
- Scalar multiplication: $x \in X, \lambda \in \mathbb{K} \implies \lambda x \in X$

and the following axioms that are satisfied for all $x, y, z \in X$ and $\lambda, \mu \in \mathbb{K}$:

1. $x + y = y + x$
2. $(x + y) + z = x + (y + z)$
3. There exists an element $0 \in X$ such that $x + 0 = x$
4. There exists an element $-x \in X$ such that $x + (-x) = 0$
5. $\lambda(\mu x) = \mu(\lambda x)$
6. $1x = x$
7. $\lambda(x + y) = \lambda x + \lambda y$
8. $(\lambda + \mu)x = \lambda x + \mu x$

Examples of linear spaces

$$\ell^p = \left\{ (x_1, x_2, x_3, \dots) : x_i \in \mathbb{K}, \quad \sum_{i=1}^{\infty} |x_i|^p < \infty \right\} \qquad (p \geq 1)$$

$$\ell^\infty = \left\{ (x_1, x_2, x_3, \dots) : x_i \in \mathbb{K}, \quad \sup |x_i| < \infty \right\}$$

$$\mathcal{C}([a, b], \mathbb{K}) = \{ f : [a, b] \to \mathbb{K} : f \text{ is continuous} \}$$

1.1.1 Quotient spaces

Definition *Equivalence relation*

$\sim$ is an **equivalence relation** on a set X if for all $x, y, z \in X$:

1. $x \sim x$
2. $x \sim y \iff y \sim x$
3. $x \sim y$ and $y \sim z \implies x \sim z$

The **equivalence class** of x is $[x] := \{ y \in X : x \sim y \}$. The set of all equivalence classes is denoted $X/\sim$
The map $\pi : X \to X/\sim$ given by $\pi(x) = [x]$ is called the **quotient map**.

Lemma

If X is a linear space and $V \subset X$ a linear subspace, then $x \sim y \iff x - y \in V$ is an equivalence relation on X.
Equivalence classes under this relation are denoted $x + V$.

Proposition *Quotient space*

Continuing from the previous lemma, $X/V := X/\sim$ becomes a linear space with:

$$(x + V) + (y + V) := (x + y) + V \qquad \lambda(x + V) := (\lambda x) + V$$

1.1.2 Linear maps

Definition *Linear map*

Let X, Y be linear spaces over $\mathbb{K}$. A map $T : \operatorname{dom} T \to Y$ is a **linear map** if for all $x, y \in X$ and $\lambda \in \mathbb{K}$:

1. the domain of T is a subspace of X
2. $T(x+y) = Tx + Ty$
3. $T(\lambda x) = \lambda T(x)$

We denote: $L(X,Y) := \{T : X \to Y : T \text{ is linear and } \operatorname{dom} T = X\}$ and $L(X) := L(X,X)$.

Definition *Sum of linear spaces*

The **sum of linear subspaces** $V, W \subset X$ is defined as:

$$V + W := \{x + y : x \in V, y \in W\}$$

We speak of a **direct sum** if $V \cap W = \{0\}$.

Definition *Projection*

$P \in L(X)$ is called a **projection** if $P^2 = P$

Lemma

If $P \in L(X)$ is a projection, then

1. $I - P$ is a projection
2. $\operatorname{ran} P = \ker(I - P)$
3. $\ker P = \operatorname{ran}(I - P)$
4. $X = \ker P + \operatorname{ran} P$ is a **direct sum**, i.e. $\operatorname{ran} P \cap \ker P = \{0\}$

Theorem

If X, Y are linear spaces, $T \in L(X,Y)$ and $V \subset \ker T$ a linear subspace, then

$$\hat{T} : X/V \to Y \qquad x + V \mapsto T(x)$$

is well-defined and linear.

Corollary

If X, Y are linear spaces and $T \in L(X,Y)$ then

$$\hat{T} : X/\ker T \to \operatorname{ran} T \qquad x + \ker T \mapsto T(x)$$

is an isomorphism.

Theorem

If X is a finite-dimensional linear space and $V \subset X$ a linear subspace, then

$$\dim X/V = \dim X - \dim V$$

Corollary

If X is a finite-dimensional linear space and $T \in L(X,Y)$, then

$$\dim \operatorname{ran} T + \dim \ker T = \dim X$$

1.1.3 Dual spaces

Definition *Dual space*

Let X be a linear space over $\mathbb{K}$. Then the **dual space** of X is defined as:

$$X' = L(X, \mathbb{K})$$

Elements of this space are called **functionals**.

Lemma

$$\dim X = n < \infty \implies \dim X' = n$$

Definition *Second dual space*

Let X be a linear space over $\mathbb{K}$. The **second dual space** of X is:

$$X'' = L(X', K)$$

We define the **natural map** as:

$$J : X \to X'' \qquad J(x)(f) = f(x) \qquad x \in X, f \in X'$$

Proposition

The natural map J is injective.

1.2 Normed linear spaces

Definition *Norm*

A **norm** on a linear space X is a real-valued function $X \to \mathbb{R}$, $x \mapsto \|x\|$ which satisfies:

1. $\|x\| \geq 0$
2. $\|x\| = 0 \iff x = 0$
3. $\|x + y\| \leq \|x\| + \|y\|$
4. $\|\lambda x\| = |\lambda| \cdot \|x\|$ for all $\lambda \in \mathbb{K}$

Note: $d(x, y) = \|x - y\|$ is a metric on X.
We abbreviate "normed linear space" by **NLS**.
If a norm does not satisfy axiom 2, then it is called a **semi-norm**.

Proposition *p-norm on $\mathbb{K}^n$*

The following are norms on $\mathbb{K}^n$:

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \qquad \|x\|_\infty = \max\{|x_i| : i \in \{1, \dots, n\}\}$$

$\|x\|_2$ is called the **Euclidean norm**. $\|x\|_p$ and $\|x\|_\infty$ are also norms on ℓ^p and ℓ^∞ respectively.

Proposition *p-norm on $\mathcal{C}([a, b], \mathbb{K})$*

The following are norms on $\mathcal{C}([a, b], \mathbb{K})$:

$$\|f\|_p = \left(\int_a^b |f(x)|^p \, \mathrm{d}x \right)^{1/p} \qquad \|f\|_\infty = \sup_{x \in [a,b]} |f(x)|$$

Lemma *Proof of the triangle inequality for $\|x\|_p$*

Young's inequality: If $1 < p < \infty$ and $a, b \geq 0$, then

$$\frac{1}{p} + \frac{1}{q} = 1 \implies ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Hölder's inequality: If $1 < p < \infty$, then

$$\frac{1}{p} + \frac{1}{q} = 1 \implies \sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |y_i|^q\right)^{1/q}$$

Minkowski's inequality: If $1 < p < \infty$, then

$$\left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |y_i|^p\right)^{1/p}$$

Lemma *Reverse triangle inequality*

If X is a normed vector space, then

$$|\|x\| - \|y\|| \leq \|x - y\| \qquad \text{for all } x, y \in X$$

1.2.1 Convergence and equivalent norms

Definition *Convergence of sequences*

A sequence (x_n) in a normed linear space X converges to $x \in X$ (denoted $x_n \to x$) w.r.t. the norm $\|\cdot\|$ if

$$\|x_n - x\| \to 0 \quad \text{as} \quad n \to \infty$$

or formally:

$$\forall \varepsilon > 0 \;\; \exists N > 0 \quad \text{such that} \quad n \geq N \implies \|x_n - x\| \leq \varepsilon$$

Lemma *Algebraic properties of limits*

$$x_n \to x \text{ in } X \implies \|x_n\| \to \|x\| \text{ in } \mathbb{R}$$
$$x_n \to x \text{ and } y_n \to y \text{ in } X \implies x_n + y_n \to x + y \text{ in } X$$
$$x_n \to x \text{ in } X \text{ and } \lambda_n \to \lambda \text{ in } \mathbb{K} \implies \lambda_n x_n \to \lambda x \text{ in } X$$

Definition *Equivalent norms*

Showing equivalence of norms is a possible exam question.

Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on X are called **equivalent** if there exist $m, M > 0$ such that

$$m\|x\|_1 \leq \|x\|_2 \leq M\|x\|_1 \qquad \text{for all } x \in X$$

Lemma

If $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent, then

$$\|x\|_1 \to 0 \iff \|x\|_2 \to 0$$

Theorem

If X is finite-dimensional, then all norms on X are equivalent.

1.3 Open, closed and compact sets

1.3.1 Open sets

Definition *Open set*

The **open** ε-**ball** centered at $x \in X$ is defined as:

$$B(x;\varepsilon) = \{y \in X : \|x - y\| < \varepsilon\}$$

$O \subset X$ is **open** if:

for all $x \in O$ there exists $\varepsilon > 0$ such that $B(x;\varepsilon) \subset O$

Proposition

If $O \subset X$ is a linear subspace and O is open, then $O = X$

1.3.2 Closed sets and closure

Definition *Distance between a point and a set*

Let $x \in X$ and $v \subset X$. The **distance** between x and V is defined as:

$$d(x, V) := \inf\{\|x - v\| : v \in V\}$$

Definition *Closure and closed sets*

Let $V \subset X$. The **closure** of V is defined as:

$$\overline{V} := \{x \in X : d(x, V) = 0\}$$

A set is **closed** if it is equal to its closure.

Proposition

$$V \subset \overline{V} \qquad \overline{V} = \overline{\overline{V}} \qquad V \subset X \text{ is closed} \iff V^c \text{ is open}$$

Lemma

If X is a NLS and $V \subset X$ is a subset, then

$$x \in \overline{V} \iff x_n \to x \text{ for some sequence } (x_n) \text{ in } V$$

Lemma

If V is a finite-dimensional subspace of a NLS, then V is closed.

Lemma

The closure of a linear subspace is a linear subspace.

Proposition

If X is a NLS and $V \subset X$ a linear subspace, then

1. $\|x + V\| := d(x, V)$ is a semi-norm on X/V
2. $\|x + V\|$ is a norm $\iff$ V is closed
3. $\|x + V\| \leq \|x\|$ for all $x \in X$

Lemma *Riesz' lemma*

If X is a NLS and $V \subset X$ is a closed linear subspace with $V \neq X$, then

$$\text{for all } 0 < \lambda < 1 \text{ there exists } x_\lambda \in X \text{ such that } \|x_\lambda\| = 1 \text{ and } \|x_\lambda - v\| > \lambda \text{ for all } v \in V$$

1.3.3 Dense, seperable and compact sets

Definition *Dense and seperable set*

Let X be a metric space.

1. A subset $E \subset X$ is called **dense** if $\overline{E} = X$
2. X is called **seperable** if it contains a countable dense subset.

Theorem

If X is a NLS, then

$$B = \{x \in X : \|x\| \leq 1\} \text{ is compact} \implies \dim X < \infty$$

Theorem

Let X be a NLS and $V \subset X$.
If X is finite-dimensional:

$$V \text{ is compact} \iff V \text{ is closed and bounded}$$

If X is infinite-dimensional:

$$V \text{ is compact} \implies V \text{ is closed and bounded}$$

1.4 Inner product spaces

Proposition *Law of cosines*

Let $\|\cdot\|$ be the Euclidean norm, $x, y \in \mathbb{R}^2$, and θ the angle between the vectors x and y.

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\|x\|\|y\|\cos(\theta)$$

$$\|x\|\|y\|\cos\theta = x_1y_1 + x_2y_2 \qquad \cos(\theta) = 0 \iff x, y \text{ are perpendicular}$$

Definition *Inner product*

Let X be a linear space over $\mathbb{K}$. A map $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{K}$ is called an **inner product** if:

1. $\langle x, x \rangle \geq 0$
2. $\langle x, x \rangle = 0 \iff x = 0$
3. $\langle \lambda x + \mu y, z \rangle = \lambda \langle x, z \rangle + \mu \langle y, z \rangle$ for all $\lambda, \mu \in \mathbb{K}$
4. $\langle x, y \rangle = \overline{\langle y, x \rangle}$ (if $\mathbb{K} = \mathbb{R}$, then $\langle x, y \rangle = \langle y, x \rangle$)

We abbreviate "inner product space" by **IPS**.

(Conjugate-)linearity of the second component

$$\text{if } \mathbb{K} = \mathbb{R}: \quad \langle x, \lambda y + \mu z \rangle = \lambda \langle x, y \rangle + \mu \langle x, z \rangle \qquad \text{if } \mathbb{K} = \mathbb{C}: \quad \langle x, \lambda y + \mu z \rangle = \overline{\lambda} \langle x, y \rangle + \overline{\mu} \langle x, z \rangle$$

Lemma *Cauchy-Schwarz inequality*

If X is an IPS, then for all $x, y \in X$:

$$|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle$$

Corollary

If X is an IPS, then $\|x\| = \sqrt{\langle x, x \rangle}$ is a norm. With this norm, we can write the Cauchy-Schwarz inequality as:

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$$

Corollary

If X is an IPS, x_n converges to x and y_n converges to y, then $\langle x_n, y_n \rangle$ converges to $\langle x, y \rangle$.
Here, the convergence is with respect to the norm induced by the inner product.

Proposition *Parallelogram law*

If $\|x\|$ is defined by an inner product, then:

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

Proposition *Parallelogram identity*

If $\|x\|$ is defined by an inner product, then:

$$4\langle x, y\rangle = \|x+y\|^2 - \|x-y\|^2 \underbrace{+i\|x+iy\|^2 - i\|x-iy\|^2}_{\text{only if } \mathbb{K}=\mathbb{C}}$$

Definition *Orthogonality*

x and y are **orthogonal** (denoted $x \perp y$) if $\langle x, y\rangle = 0$.

Theorem *Pythagorean theorem*

$$x \perp y \implies \|x+y\|^2 = \|x\|^2 + \|y\|^2$$

1.4.1 Best approximations

Lemma

If X is an IPS and $V \subset X$ a subset, then the **orthogonal complement** of V defined by

$$V^\perp = \{x \in X : \langle x, v\rangle = 0 \text{ for all } v \in V\}$$

is a closed linear subspace.

Definition

Let X be a NLS and $V \subset X$ a subset. $v_0 \in V$ is called a **best approximation** of $x \in X$ if

$$\|x - v_0\| = d(x, V) := \inf\{\|x - v\| : v \in V\}$$

Lemma

Let X be an IPS and $V \subset X$ a linear subspace. If $x \in X$ and $v_0 \in V$, then

$$\|x - v_0\| = d(x, V) \iff x - v_0 \in V^\perp$$

Lemma

If X is an IPS and $V \subset X$ is a finite-dimensional linear subspace,
then for all $x \in X$ there exists a unique best approximation $v_0 \in V$.

Theorem *Computation of the best approximation in a finite-dimensional space*

Let X be an IPS, $V \subset X$ a finite-dimensional linear subspace, and $\{e_1, \ldots, e_n\}$ an orthonormal basis of V.
Then $v_0 = c_1e_1 + \ldots + c_ne_n$ is the unique best approximation of x, with $c_i = \langle x, e_1\rangle$.

1.4.2 Orthonormal systems

Definition *Orthonormal set*

If X is an IPS, then $\{e_i : i \in I\} \subset X$ is called an **orthonormal set** if $\begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

Proposition

Orthonormal vectors are linearly independent

Algorithm *Gram-Schmidt procedure*

Let X be an IPS and let $f_1, \dots, f_n$ be linearly independent.
There exist orthonormal vectors $e_1, \dots, e_n$ such that $\operatorname{span}\{e_1, \dots, e_k\} = \operatorname{span}\{f_1, \dots, f_k\}$ for all $k \in \{1, \dots, n\}$
These vectors e_i are constructed as follows:

$$e_1 = \frac{f_1}{\|f_1\|} \qquad \tilde{e}_2 = f_2 - \langle f_2, e_1 \rangle e_1 \qquad e_2 = \frac{\tilde{e}_2}{\|\tilde{e}_2\|} \qquad \tilde{e}_{k+1} = f_{k+1} - \sum_{i=1}^{k} \langle f_{k+1}, e_i \rangle e_i \qquad e_{k+1} = \frac{\tilde{e}_{k+1}}{\|\tilde{e}_{k+1}\|}$$

1.5 Banach and Hilbert spaces

1.5.1 Banach spaces

Definition *Cauchy sequence*

(x_n) is a **Cauchy sequence** in a normed linear space X if:

$$\forall \varepsilon > 0 \;\; \exists N > 0 \quad \text{such that} \quad n, m \geq N \implies \|x_n - x_m\| \leq \varepsilon$$

Proposition

Every convergent sequence is a Cauchy sequence.

Definition *Banach space*

A normed linear space X is called a **Banach space** or **complete space** if every Cauchy sequence in X converges.

Proposition

Every finite-dimensional normed linear space is a Banach space.

Theorem

The following are Banach spaces:

1. ℓ^p with the norm $\|x\|_p$
2. ℓ^∞ with the norm $\|x\|_\infty$
3. $\mathcal{C}([a,b], \mathbb{K})$ with the norm $\|f\|_\infty$

Note: $\mathcal{C}([a,b], \mathbb{K})$ is not complete with the norm $\|f\|_p$.

Proposition

If X is a NLS and $V \subset X$ is a linear subspace, then:

1. X Banach and V closed $\implies$ V Banach
2. V Banach $\implies$ V closed in X

1.5.2 Hilbert spaces

Definition *Hilbert space*

A **Hilbert space** is a Banach space of which the norm comes from an inner product.

Examples of Hilbert spaces

These are the only examples of separable Hilbert spaces up to isomorphism:

$$\mathbb{K}^n \quad \langle x, y \rangle := \sum_{i=1}^{n} x_i \overline{y_i} \quad \|x\| = \sqrt{\langle x, x \rangle} \qquad\qquad \ell^2 \quad \langle x, y \rangle := \sum_{i=1}^{\infty} x_i \overline{y_i} \quad \|x\| = \sqrt{\langle x, x \rangle}$$

Definition *Convex set*

A set $V \subset X$ is called **convex** if:

$$x, y \in V \implies \lambda x + (1-\lambda)y \in V \quad \text{for all } \lambda \in [0,1]$$

Theorem *Existence and uniqueness of best approximations*

If X is a Hilbert space and $V \subset X$ is a nonempty, closed and convex subset, then

$$\text{for all } x \in X \text{ there exists a unique } v \in V \text{ such that } \|x - v\| = d(x, V)$$

Theorem *Orthogonal decompositions*

If X is a Hilbert space and $V \subset X$ is a closed linear subspace, then

$$\text{for all } x \in X \text{ there exist unique } v \in V,\, w \in V^{\perp} \text{ such that } x = v + w$$

Note: V and $V^{\perp}$ are Hilbert spaces, so we can again decompose v and w.

1.5.3 Completions

Theorem *Completion theorem*

Let X be a NLS. There exists a Banach space $\tilde{X}$ and a linear map $\iota : X \to \tilde{X}$ such that

1. X and $\iota(X)$ are isometrically isomorphic
2. $\iota(X)$ is dense in $\tilde{X}$

Definition $L^p(a,b)$

$L^p(a,b)$ is the completion of $\mathcal{C}([a,b],\mathbb{K})$ with respect to the norm $\|f\|_p$.

Proposition

$L^2(a,b)$ is a Hilbert space isomorphic to ℓ^2 with the inner product $\displaystyle\int_a^b f(t)\overline{g(t)}\,\mathrm{d}t$

1.6 Orthonormal bases

Definition *Hamel basis*

A subset $B \subset X$ is called a **Hamel basis** if B is a set of linearly independent vectors and $X = \mathrm{span}(B)$.
This definition works if X is a finite-dimensional space and does not work for general separable Banach spaces.

Lemma *Bessel's inequality*

If X is an inner product space and $\{e_k : k \in \mathbb{N}\}$ is an orthonormal set, then

$$\sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2 \leq \|x\|^2 \qquad \text{for all } x \in X$$

In particular, the series on the left converges.

Theorem

If X is a Hilbert space and $\{e_k : k \in \mathbb{N}\}$ is an orthonormal set, then

$$\sum_{k=1}^{\infty} \lambda_k e_k \text{ converges in } X \iff \sum_{k=1}^{\infty} |\lambda_k|^2 < \infty \implies \left\|\sum_{k=1}^{\infty} \lambda_k e_k\right\|^2 = \sum_{k=1}^{\infty} |\lambda_k|^2$$

Corollary

If X is a Hilbert space and $\{e_k : k \in \mathbb{N}\}$ is an orthonormal set, then

$$\sum_{k=1}^{\infty} \langle x, e_k \rangle e_k \text{ converges for all } x \in X$$

Definition *Orthonormal basis*

Let X be a Hilbert space. The orthonormal set $\{e_k : k \in \mathbb{N}\}$ is called an **orthonormal basis** for X if

$$\overline{\operatorname{span}\{e_k : k \in \mathbb{N}\}} = X$$

Theorem

Let X be a Hilbert space and $\{e_k : k \in \mathbb{N}\}$ an orthonormal set. The following are equivalent:

1. $\{e_k : k \in \mathbb{N}\}$ form an orthonormal basis for X
2. $\{e_k : k \in \mathbb{N}\}^{\perp} = \{0\}$
3. $\|x\|^2 = \sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2$ for all $x \in X$
4. $x = \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k$ for all $x \in X$

Theorem

If X is an infinite-dimensional Hilbert space, then

$$X \text{ has an orthonormal basis} \iff X \text{ is seperable}$$

Corollary

All separable infinite-dimensional Hilbert spaces are isomorphic with ℓ^2.

1.6.1 Fourier series

Proposition

The functions $1, \sin(kx), \cos(kx)$ for $k \in \mathbb{N}$ form an orthogonal basis for L^2.

Theorem *Fourier series*

Any $f \in L^2(-\pi, \pi)$ can be written as a **Fourier series**:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos(kx) + b_k \sin(kx)) \qquad a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) \, \mathrm{d}x \qquad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) \, \mathrm{d}x$$

The Fourier series converges with respect to the L^2 norm: $\lim_{n \to \infty} \int_{-\pi}^{\pi} |f(x) - s_n(x)|^2 \, \mathrm{d}x = 0$

Proposition

The functions $b_k(x) = x^k$ with $k \in \mathbb{N}$ are linearly independent, and their span is dense in $\mathcal{C}([-1, 1], \mathbb{K})$ and $L^2(-1, 1)$. They are however not orthogonal.

2 Linear operators

2.1 Bounded and invertible linear operators

2.1.1 Continuous operators

Definition *Continuous linear operator*

Let X, Y be normed linear spaces and $T \in L(X,Y)$. T is called **continuous** at $x_0 \in X$ if

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \text{such that} \quad \|x_0 - x\| < \delta \implies \|T(x_0) - T(x)\| < \varepsilon \quad \text{for all } x \in X$$

Lemma

$$\text{continuity at } 0 \iff \text{continuity at every } x_0 \in X$$

Lemma

$$\text{continuity at } 0 \iff \text{there exists } c > 0 \text{ such that } \|Tx\| \le c\|x\| \text{ for all } x \in X$$

2.1.2 Bounded operators

Definition *Bounded linear operator*

Let X, Y linear spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ respectively, and $T \in L(X,Y)$.
T is called **bounded** if there exists $c > 0$ such that

$$\|Tx\|_Y \le c\|x\|_X$$

Note: this does not imply $\|Tx\| \le c$ for all $x \in X$.

Definition *Operator norm*

Computing an operator norm is a possible exam question.
Let X, Y be normed linear spaces and let $T \in L(X,Y)$. If T is bounded, we define its **operator norm** by

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|_Y}{\|x\|_X}$$

2.1.3 Spaces of bounded operators

Definition $B(X,Y)$

$$B(X,Y) = \{T \in L(X,Y) : T \text{ bounded}\}$$

Lemma

$B(X,Y)$ is a linear space, and the operator norm $\|T\|$ is a norm on $B(X,Y)$

Lemma

If X and Y are normed linear spaces and $T \in B(X,Y)$, then

$$\|Tx\| \le \|T\|\|x\| \text{ for all } x \in X$$

Lemma

Let X, Y, Z be normed linear spaces, $T \in B(X,Y)$ and $S \in B(Y,Z)$. Then

$$ST \in B(X,Z) \qquad \|ST\| \le \|S\|\|T\|$$

Lemma

If $T_n \in B(X,Y)$ and $S_n \in B(Y,Z)$ for all $n \in \mathbb{N}$, then

$$T_n \to T \text{ and } S_n \to S \implies S_nT_n \to ST$$

Note: $T_n \to T$ means $\|T_n - T\|_{B(X,Y)} \to 0$

Theorem

If X, Y are normed linear spaces, then

$$Y \text{ Banach} \implies B(X,Y) \text{ Banach}$$

2.1.4 Invertible operators

Definition

$T \in B(X,Y)$ is called **invertible** if

1. $T : X \to Y$ is a bijection
2. $T^{-1} \in B(Y,X)$

Note: (1) does not imply (2).

Lemma

$$T \in B(X,Y) \text{ invertible} \iff \text{there exists } S \in B(Y,X) \text{ such that } ST = I_X \text{ and } TS = I_Y$$

Theorem *Computation of* $(I-T)^{-1}$

If X is Banach and $T \in B(X,X)$, then

$$\sum_{k=0}^{\infty} \|T^k\| \leq \infty \implies (I-T)^{-1} = \sum_{k=0}^{\infty} T^k \in B(X)$$

In particular, this works when $\|T\| < 1$, since $\|T^k\| \leq \|T\|^k$

2.2 Compact operators

2.2.1 Eigenspaces

Definition *Eigenspace*

Let T be a linear operator with some eigenvalue λ. The **eigenspace** is defined as

$$E_\lambda = \ker(T - \lambda I) = \{x \in X : Tx = \lambda x\}$$

Proposition

$$\dim E_\lambda < \infty \implies B_\lambda := \{x \in E_\lambda : \|x\| \leq 1\} \text{ is compact} \implies T(B_\lambda) \text{ is compact}$$

Definition *Compact operator*

$T \in L(X,Y)$ is **compact** if

$$V \text{ is a bounded set} \implies T(V) \text{ is relatively compact}$$

A set is **relatively compact** if its closure is compact.

Lemma

$$T \text{ compact} \implies T \text{ bounded}$$

Lemma

The following are equivalent:

1. $T \in L(X,Y)$ is compact
2. (x_n) is a bounded sequence $\implies$ (Tx_n) has a convergent subsequence

Lemma

If $T \in B(X,Y)$ and the range of T is finite-dimensional, then T is compact.

2.2.2 Spaces of compact operators

Definition $K(X,Y)$

$$K(X,Y) = \{T \in L(X,Y) : T \text{ is compact}\}$$

Lemma

1. $K(X,Y)$ is a linear subspace of $B(X,Y)$
2. If $T \in B(X,Y)$ and $S \in B(Y,Z)$, then

$$T \text{ or } S \text{ is compact} \implies ST \text{ is compact}$$

Theorem

If X is a normed linear space and Y is Banach, then $K(X,Y)$ is closed in $B(X,Y)$.

Proving compactness of an operator

Compactness of an operator $T(X,Y)$ with Y Banach can be proven as follows:

1. Construct a bounded sequence T_n converging to T, where $\operatorname{ran} T_n$ is finite-dimensional for all finite n.
2. Since $\operatorname{ran} T_n$ is finite-dimensional, and T_n is bounded, each T_n is compact.
3. Show that T_n converges to T.
4. Since $K(X,Y)$ is closed by the previous theorem, T is compact.

2.2.3 Equicontinuity

Definition *Equicontinuity*

A set $V \subset \mathcal{C}([a,b],\mathbb{K})$ is called **equicontinuous** if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|x-y| < \delta \implies |f(x) - f(y)| < \varepsilon \quad \text{for all } x,y \in [a,b],\ f \in V$$

i.e. each $f \in V$ is uniformly continuous on $[a,b]$, and for a given ε the same δ works for all $f \in V$.

Theorem *Arzelà-Ascoli theorem*

If $V \subset \mathcal{C}([a,b],\mathbb{K})$, then

$$V \text{ relatively compact} \iff V \text{ bounded and equicontinuous}$$

Theorem *Integral operators*

Let $G : [a,b] \times [a,b] \to \mathbb{K}$ be continuous.
Operators $T : \mathcal{C}([a,b],\mathbb{K}) \to \mathcal{C}([a,b],\mathbb{K})$ of the following forms are compact:

Fredholm operator: $Tf(x) = \int_a^b G(x,y) f(y)\,\mathrm{d}y$ **Volterra operator**: $Tf(x) = \int_a^x G(x,y) f(y)\,\mathrm{d}y$

2.3 Meager sets and semi-norms

2.3.1 Nowhere dense sets

Definition *Interior*

Let M be a subset of a metric space (X, d). The **interior** of M, denoted $\operatorname{int} M$, is the union of all open sets in M. In other words, the interior of M is the largest open set contained in M.

Definition *Nowhere dense set*

A subset M of a metric space (X, d) is called **nowhere dense** if $\operatorname{int}(\overline{M}) = \varnothing$

Proposition

Let X be a NLS and $V \subset X$ a closed linear subspace. Then if $V \neq X$, V is nowhere dense.

Lemma

If $M \subset X$ is nowhere dense, then:

$$B(x; \varepsilon) \cap (\overline{M})^c \neq \varnothing \quad \forall x \in X \quad \forall \varepsilon > 0$$

2.3.2 Baire's theorem

Definition *Meager set*

A subset $M \subset X$ of a metric space is **meager** if it can be written as a countable union of nowhere dense sets.

Theorem *Baire's theorem*

If (X, d) is a complete metric space, then

$$O \subset X \text{ nonempty and open} \quad \Longrightarrow O \text{ nonmeager}$$

Proposition

Let $\|\cdot\|$ be any norm on

$$\mathcal{P} = \{p : \mathbb{K} \to \mathbb{K} : p \text{ is a polynomial } \}$$

Then $\mathcal{P}$ is not a Banach space.

Proposition

If X is Banach and infinite-dimensional, then there is no countable set (Hamel basis) that spans X.

2.3.3 Semi-norms

Definition *Semi-norm*

A **semi-norm** on X is a map $p : X \to [0, \infty)$ such that for all $x, y \in X$, $\lambda \in \mathbb{K}$

- $p(x + y) \leq p(x) + p(y)$
- $p(\lambda x) = |\lambda| p(x)$

A semi-norm is a norm without the property $x = 0 \iff \|x\| = 0$ (positive definiteness).

Proposition

If Y is a NLS and $T \in L(X, Y)$, then $p(x) = \|Tx\|$ is a semi-norm on X.
If T is injective, then $\|Tx\|$ is a norm.

Definition *Bounded semi-norm*

If X is a NLS, then a semi-norm p on X is **bounded** if there exists $c > 0$ such that

$$p(x) \leq c\|x\| \quad \text{for all } x \in X$$

Proposition

If $p(x) = \|Tx\|$, then

$$T \text{ is bounded} \iff p \text{ is bounded}$$

Lemma

If a semi-norm $p : X \to [0, \infty)$ is bounded, then

$$|p(x) - p(y)| \leq c\|x - y\| \quad \forall x, y \in X$$

Hence, $x_n \to x \implies p(x_n) \to p(x)$.

2.3.4 Countable subadditivity

Lemma

Bounded semi-norms are **countably subadditive**:

$$\sum_{j=1}^{\infty} x_j \text{ convergent} \implies p\left(\sum_{j=1}^{\infty} x_j\right) \leq \sum_{j=1}^{\infty} p(x_j)$$

Lemma *Zabreĭko's lemma*

Assume

- X is a Banach space
- $p : X \to [0, \infty)$ is a semi-norm
- p is countably subadditive:

$$\sum_{j=1}^{\infty} x_j \text{ convergent} \implies p\left(\sum_{j=1}^{\infty} x_j\right) \leq \sum_{j=1}^{\infty} p(x_j)$$

Then p is bounded.

2.4 Open and closed operators

Note

On the exam, you will have to use at least one of the following:

- Open mapping theorem
- Closed graph theorem
- Uniform boundedness principle

2.4.1 Open mapping theorem

Theorem *Open mapping theorem*

If X, Y are Banach spaces, and $T \in B(X, Y)$ is surjective, then T is an **open map**:

$$O \subset X \text{ open} \implies T(O) \subset Y \text{ open}$$

Corollary

If X, Y are Banach spaces, and $T \in B(X, Y)$ is bijective, then

$$T^{-1} \in B(Y, X)$$

Theorem

Assume X, Y are Banach and $T \in B(X,Y)$. The following are equivalent:

1. There exists $c > 0$ such that $\|Tx\| \geq c\|x\|$ for all $x \in X$.
2. T is injective and $\operatorname{ran} T$ is closed.

2.4.2 Closed graph theorem

Definition *Graphs and closed operators*

Let X, Y be normed linear spaces and $V \subset X$ a linear subspace.

- The **graph** of $T \in L(V,Y)$ is defined as

$$G(T) := \{(x, Tx) : x \in V\} \subset X \times Y$$

  Note: $G(X,Y)$ is a linear subspace of $X \times Y$.
- The operator T is called **closed** if $G(T)$ is closed in $X \times Y$

Lemma

$$(x, y) \in \overline{G(T)} \iff \text{there exists a sequence } (x_n) \text{ such that } x_n \to x \text{ and } Tx_n \to y$$

Lemma

If X, Y are normed linear spaces and $V \subset X$ a closed linear subspace, then

$$T \in B(V,Y) \implies T \text{ is closed}$$

Theorem *Closed graph theorem*

If X, Y are Banach spaces and $V \subset X$ a closed linear subspace, then

$$T \text{ is closed} \implies T \in B(V,Y)$$

2.4.3 Uniform boundedness principle

Theorem *Uniform boundedness principle*

Assume X is Banach and Y is a NLS. For any set $F \subset B(X,Y)$ we have

$$\sup_{T \in F} \|Tx\| < \infty \text{ for all } x \in X \implies \sup_{T \in F} \|T\| < \infty$$

Corollary

Let X be a Banach space and Y be a NLS.
Let (T_n) be a sequence in $B(X,Y)$ such that $T_n x$ converges for all $x \in X$.
If $T \in L(X,Y)$ is defined pointwise by

$$Tx := \lim_{n \to \infty} T_n x$$

then $T \in B(X,Y)$.

2.5 Spectra of linear operators

Definition *Spectrum*

For X Banach (so the Open Mapping Theorem holds) and $T \in B(X)$ we define the

- **resolvent set**: $\rho(T) = \{\lambda \in \mathbb{K} : (T - \lambda I)^{-1} \in B(X)\}$
- **resolvent operator**: $R(\lambda) = (T - \lambda I)^{-1} \quad \lambda \in \rho(T)$
- **spectrum**: $\sigma(T) = \mathbb{K} \setminus \rho(T)$

Note: λ will be used as a shorthand notation for λI from now on.
There will be an exam question about spectra. (possibly of the form "determine the spectrum of this operator")

Definition *Eigenvalues*

If $T \in B(X)$, then

- $\lambda \in \mathbb{K}$ is called an **eigenvalue** of T if there exists $x \neq 0$ such that $(T - \lambda)x = 0$ (or alternatively $Tx = \lambda x$)
- $\ker(T - \lambda)$ is called the associated **eigenspace**.
- Nonzero elements of the eigenspace are called **eigenvectors**.
- The set of eigenvalues of T, denoted $\sigma_p(T)$, is called the **point spectrum** of T.

Lemma

Assume X is Banach and $T \in B(X)$. If $|\lambda| > \|T\|$, then

$$\lambda \in \rho(T) \quad \text{and} \quad R(\lambda) = -\sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}$$

Corollary

If $\lambda \in \sigma(T)$, then $|\lambda| \leq \|T\|$.

Lemma

Assume X is Banach and $T \in B(X)$. If $\mu \in \rho(T)$ and $|\lambda - \mu| < \frac{1}{\|R(\mu)\|}$, then

$$\lambda \in \rho(T) \quad \text{and} \quad R(\lambda) = \sum_{n=0}^{\infty} (\lambda - \mu)^n R(\mu)^{n+1}$$

Corollary

$\rho(T)$ is open and $\sigma(T)$ is closed.

Definition

$\lambda \in \mathbb{K}$ is called an **approximate eigenvalue** of T if there exists a sequence (x_n) such that

$$\|x_n\| = 1 \text{ for all } n \in \mathbb{N} \quad \text{and} \quad (T - \lambda)x_n \to 0$$

Proposition *Characterization of the resolvent set*

If X is Banach and $T \in B(X)$, then $\lambda \in \rho(T)$ if and only if:

$$\operatorname{ran}(T - \lambda) \text{ dense in } X \quad \text{and} \quad \|(T - \lambda)x\| \geq c\|x\| \quad \text{for all } x \in X$$

Corollary *Characterization of the spectrum*

If X is Banach and $T \in B(X)$, then $\lambda \in \sigma(T)$ if and only if:

$$\operatorname{ran}(T - \lambda) \text{ not dense in } X \quad \text{or} \quad \lambda \text{ is an approximate eigenvalue}$$

Theorem *Spectral mapping theorem*

Assume X is Banach over $\mathbb{K} = \mathbb{C}$ and $T \in B(X)$. For any polynomial $p : \mathbb{K} \to \mathbb{K}$ we have

$$\sigma(p(T)) = \{p(\lambda) : \lambda \in \sigma(T)\}$$

Theorem *Spectral theorem for compact operators*

If X is Banach and $T \in K(X)$, then

1. For every $\varepsilon > 0$, the number of eigenvalues λ of T with $|\lambda| > \varepsilon$ is finite.
2. If $\lambda \neq 0$ is an eigenvalue of T, then $\dim \ker(T - \lambda) < \infty$
3. If $\dim X = \infty$, then $0 \in \sigma(T)$.

2.6 Adjoint operators

2.6.1 Adjoints in $\mathbb{K}^n$

Definition *Adjoint of a matrix*

The **adjoint** or **conjugate transpose** of a $n \times n$ matrix A over K is defined as:

$$A^* = (\overline{A})^\top$$

Proposition

For the standard inner product on $\mathbb{K}^n$ we have $\langle Ax, y\rangle = \langle x, A^*y\rangle$ for all $x, y \in \mathbb{K}^n$

Definition *Self-adjoint matrix*

An $n \times n$ matrix is called **self-adjoint** if $A^* = A$.

Proposition

Self-adjoint matrices are diagonalizable and they have real eigenvalues.

2.6.2 Dual spaces

Definition *Dual of a normed linear space*

Let X be a normed linear space. Then the **dual space** of X is defined as $X' = B(X, \mathbb{K})$ with the following norm:

$$\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|}$$

Lemma

Let X be an inner product space and $y \in X$.
The map $f : X \to \mathbb{K}$ defined by $f(x) = \langle x, y\rangle$ belongs to X' and

$$\|f\| = \|y\|$$

Theorem *Riesz-Fréchet theorem*

Assume X is a Hilbert space. For each $f \in X'$ there exists a unique $y \in X$ such that

$$f(x) = \langle x, y\rangle \text{ for all } x \in X$$

2.6.3 Adjoints in Hilbert spaces

Theorem *Existence of adjoints*

Let X, Y be Hilbert spaces and $T \in B(X, Y)$.
There exists a unique **adjoint operator** $T^* \in B(Y, X)$ such that

- $\langle Tx, y\rangle_Y = \langle x, T^*y\rangle_X$ for all $x \in X$ and $y \in Y$
- $\|T^*\| \leq \|T\|$

Lemma *Properties of adjoints*

Let X, Y be Hilbert spaces and $T \in B(X, Y)$.

1. $(T^*)^* = T$
2. $\|T^*\| = \|T\|$
3. $\|T^*T\| = \|T\|^2$

Let X, Y, Z be Hilbert spaces.

1. $T, S \in B(X, Y) \implies (\lambda T + \mu S)^* = \overline{\lambda} T^* + \overline{\mu} S^*$
2. $T \in B(X, Y)$ and $S \in (Y, Z) \implies (ST)^* = T^* S^*$
3. $T \in K(X, Y) \implies T^* \in K(Y, X)$

If T is invertible, then T^* is invertible and

$$(T^*)^{-1} = (T^{-1})^*$$

Lemma *Spectrum of adjoints*

If $T \in B(X)$, then

$$\rho(T^*) = \overline{\rho(T)} \qquad \sigma(T^*) = \overline{\sigma(T)}$$

Corollary

If T is self-adjoint, then $\sigma(T) \subseteq \mathbb{R}$

Lemma *Range/kernel orthogonality*

For $T \in B(X)$ and $\lambda \in \mathbb{K}$, we have

$$(\operatorname{ran}(T - \lambda))^\perp = \ker(T^* - \overline{\lambda}) \qquad (\operatorname{ran}(T^* - \overline{\lambda}))^\perp = \ker(T - \lambda)$$

Corollary

Let $T \in B(X)$ and $\lambda \in \mathbb{K}$. We have the following orthogonal decompositions:

$$X = (\operatorname{ran}(T - \lambda))^\perp \oplus \ker(T^* - \overline{\lambda}) \qquad X = (\operatorname{ran}(T^* - \overline{\lambda}))^\perp \oplus \ker(T - \lambda)$$

2.6.4 Normal operators

Definition *Normal and unitary operators*

$T \in B(X)$ is **normal** if $TT^* = T^*T$.
$T \in B(X, Y)$ is **unitary** if $T^*T = I_X$ and $TT^* = I_Y$.

Lemma

If $T \in B(X)$ is normal, then $\|Tx\| = \|T^*x\|$ for all $x \in X$.

Corollary

If $T \in B(X)$ is normal, then for all $\lambda \in \mathbb{K}$

$$\ker(T^* - \overline{\lambda}) = \ker(T - \lambda)$$

Lemma *Resolvent set of a normal operator*

If $T \in B(X)$ is normal, then

$$\rho(T) = \{\lambda \in \mathbb{K} : \text{ there exists } c > 0 \text{ such that } \|(T - \lambda)x\| \geq c\|x\| \quad \forall x \in X\}$$

Corollary *Spectrum of a normal operator*

If $T \in B(X)$ is normal, then

$$\sigma(T) = \{\lambda \in \mathbb{K} : \text{ there exists } (x_n) \text{ such that } \|x_n\| = 1 \text{ and } (T - \lambda)x_n \to 0\}$$

i.e. the spectrum is equal to the set of approximate eigenvalues.

Lemma

If $T \in B(X)$ is normal and $\lambda \neq \mu$, then

$$Tx = \lambda x \text{ and } Ty = \mu y \implies \langle x, y\rangle = 0$$

2.7 Self-adjoint operators

Definition *Self-adjoint operator*

$T \in B(X)$ is **self-adjoint** if $T = T^*$

Lemma

Let X be a Hilbert space with $\mathbb{K} = \mathbb{C}$.

$$T \text{ is self-adjoint} \iff \langle Tx, x\rangle \in \mathbb{R} \text{ for all } X$$

2.7.1 Nonnegative operators

Definition *Nonnegative operator*

$T \in B(X)$ is **nonnegative**, denoted $T \geq 0$, if $\langle Tx, x\rangle \geq 0$ for all $x \in X$.

Corollary

If $\mathbb{K} = \mathbb{C}$, then nonnegative operators are self-adjoint.

Lemma

If P is an orthogonal projection, then $P \geq 0$.

Lemma

If T is nonnegative, then

$$\|Tx\|^2 \leq \|T\|\langle Tx, x\rangle \qquad \text{for all } x \in X$$

Lemma

If T is nonnegative, then

$$\|T\| = \sup_{\|x\|=1} \langle Tx, x\rangle$$

2.7.2 Spectra of self-adjoint operators

Definition *a and b*

For a self-adjoint operator $T \in B(X)$ we define

$$a := \inf_{\|x\|=1} \langle Tx, x \rangle \qquad b := \sup_{\|x\|=1} \langle Tx, x \rangle$$

Lemma

$$T - aI \geq 0 \qquad bI - T \geq 0$$

Theorem

If T is self-adjoint, then

1. $Tx = \lambda x$ and $Ty = \mu y$ with $\lambda \neq \mu$ implies $\langle x, y \rangle = 0$
2. $\sigma(T)$ only contains approximate eigenvalues
3. $\sigma(T) \subset [a, b]$
4. $a, b \in \sigma(T)$

Note: 1) and 2) follow from T being normal.

Theorem

If T is self-adjoint, then

$$\|T\| = \sup_{\|x\|=1} |\langle Tx, x \rangle| = \max\{|a|, |b|\}$$

2.7.3 Eigenvalues of compact self-adjoint operators

Proposition

If T is compact and self-adjoint, then $\|T\|$ or $-\|T\|$ (or both) is an eigenvalue.

Lemma

If V is a linear subspace of X and $T \in B(X)$, then

$$T(V) \subset V \implies T^*(V^\perp) \subset V^\perp$$

Theorem *Diagonalization theorem*

If X is a Hilbert space and $T \in B(X)$ is self-adjoint and compact, then there exist:

- countably many real eigenvalues λ_i
- countably many orthonormal eigenvectors e_i

such that

$$Tx = \sum_i \lambda_i \langle x, e_i \rangle e_i$$

2.8 Hahn-Banach theorem

2.8.1 Ordering of sets

Definition *Partial order*

Assume X is a nonempty set. $\preceq$ is called a **partial order** on X if

1. $x \preceq x$ for all $x \in X$
2. $x \preceq y$ and $y \preceq x \implies x = y$
3. $x \preceq y$ and $y \preceq z \implies x \preceq z$

$\preceq$ is called a **total order** if for all $x, y \in X$ we have

$$x \preceq y \quad \text{or} \quad y \preceq x$$

Definition *Upper bound and maximal element*

If $\preceq$ is a partial order on X and $V \subset X$, then $y \in X$ is called

- an **upper bound** for V if $x \preceq y$ for all $x \in V$
- a **maximal element** of X if $y \preceq x \implies y = x$

Lemma *Zorn's lemma*

Let $X \neq \varnothing$ be partially ordered.
If every totally ordered subset of X has an upper bound in X, then X has a maximal element.

2.8.2 Hahn-Banach theorem

Theorem *Hahn-Banach theorem*

Assume that

- X is a linear space
- $V \subset X$ is a **proper linear subspace** ($V \neq \{0\}$ and $V \neq X$)
- $p : X \to [0, \infty)$ is a semi-norm
- $f \in L(V, \mathbb{K})$ satisfies the bound

$$|f(x)| \leq p(x) \text{ for all } x \in V$$

Then there exists $F \in L(X, \mathbb{K})$ such that

$$F \restriction V = f \qquad |F(x)| \leq p(x) \text{ for all } x \in X$$

(Note: $F \restriction V$ denotes "F restricted to V".)

Theorem *Hahn-Banach theorem for normed linear spaces*

If X is a normed linear space and $V \subset X$ is a linear subspace, then for all $f \in V'$ there exists $F \in X'$ such that

$$F \restriction V = f \qquad \|F\| = \|f\|$$

Note: $\|f\|$ is the operator norm on V, and $\|F\|$ is the operator norm on X.
Exercises will only use this version of the Hahn-Banach theorem, you don't have to remember the other one.

2.8.3 Applications

Proposition

If X is a normed linear space and $x, y \in X$ are distinct, then there exists $f \in X'$ such that

$$\|f\| = 1 \qquad f(x) \neq f(y)$$

Proposition

If X is a normed linear space and $x_0 \in X$ is nonzero, then there exists $f \in X'$ such that

$$\|f\| = 1 \qquad f(x_0) = \|x_0\|$$

Corollary

The norm of any nonzero $x_0 \in X$ can be written as

$$\|x_0\| = \sup\{|f(x_0)| : f \in X', \|f\| = 1\}$$

Proposition

If X is a normed linear space and $V \subset X$ is a finite-dimensional linear subspace, then there exists $P \in B(X)$ such that

$$P^2 = P \qquad \operatorname{ran} P = V$$

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